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Boolean Algebra

Task 2

Another standard algebraic law for multiplication is associativity:

(xy)z = x(yz) for all x, y, z

Provide an example of classes x, y and z to illustrate why this law also holds for literal symbols representing classes. Then explain how we know that Boole assumed associativity for multiplication in his section 6 above, even though he did not explicitly mention it.

The classes X, Y, and Z all represent different objects and properties that are unique from each other. In other words they are all mutually disjoint. Applying the property z to the objects resulted by xy gives the same representation as when the property x is appended to the resulted yz product. In literal terms we can assign objects to each variable. X could represent elephants, y could represent the color green, and the letter z can represent the characteristic tall. Any combination of these three characteristics results in the same definitive product. For example, all green elephants that are tall is the same distinction as all green and tall elephants. The order that the characteristic is assignment doesn’t matter. So, no matter the grouping of the characteristic, the end result will always be the same and (xy)z = x(yz) = (xz)y.

Task 4

Consider, for instance, the following expressions:

(I) Infants and teenagers (III) Lying or confused

(II) Dancers and singers (IV) Conqueror of Gaul or first emperor of Rome

For which of these is the conjunction (or, and) used in the exclusive sense specified by Boole? That is, is there an implication in standard language usage that a particular individual under discussion will belong to at most one of the named classes, but not both classes simultaneously?

1. Infants and teenagers is used in the exclusive sense that is determined by Boole. This is the case since no element of either set Infants or Teenagers share anything in common in terms of age. No infant is as old as a teenager and no teenager is as young as an infant. Their age clearly separates them into exclusive classes.
2. Dancers and singers doesn’t advocate for the exclusive sense specified by Boole, because a dancer could also identify themselves as a singer while a singer could identify themselves as a dancer as well. This is the case if each individual is entitled to more than one class. However, if any person is only distinguished by only one class or the other than the conjunction is used in the exclusive sense specified by Boole.
3. The conjunction Lying or confused doesn’t conform to the exclusive sense as well since a particular individual may be both lying and confused at the same time. More specifically they could be lying because they are confused. However, there are times when a particular individual may be either one or the other exclusively.
4. The conjunction the conqueror of Gaul or first emperor of Rome does conform to the exclusive sense defined by Boole since each of the titles on either side of the conjunction belong to different men. This distinction allows for it to be exclusive.

Task 6

Later in his section 11, Boole justified the distributivity law,

z(x + y) = zx + zy, (4)

by letting x represent “men,” y represent “women,” and z represent “European.” Describe, in English, the meaning of each side of the equation for this particular example. Is this is a convincing justification for the law?

In the parenthesis in the right side of the equals sign, we see two classes x and y. These two classes are mutually disjoint, meaning they have nothing in common. In this case they are separated by men and women. To the right of the parentheses we have the letter z representing the characteristic European. This characteristic is the one thing that both the men and women share in common. So, the z is written outside the parenthesis as a multiplication operation where it will apply to both x and y that are inside the parenthesis. This implies that both classes x and y share the characteristic z.

On the right side of the equals sign we see that the two separate classes x and y share one characteristic and that is the characteristic z. Since the z is written beside both x and y, both of them have that characteristic. It implies the same thing as what the expression to the left of the equal sign does. After the z is distributed to the x and y, we clearly see that both of the classes, men and women, share the characteristic European.

Yes, this is a convincing justification of the distributive law. It explains that two disjoint classes can be redefined to have new characteristic that applies to both of them.

Task 8

Now consider Boole’s equations (7) and (8) in his section 13 on the previous page, and recall his earlier restriction that “+” can only be applied to classes which share no members. Suppose we were to drop this restriction, and let y represent men, z represent doctors, and x = y + z as in Boole’s equation (7). Can we still deduce Boole’s equation (8) in this case? That is, does x − z = y? Explain.

x = y + z. (7)

x − z = y, (8)

No, x minus z does not equal y. In the case that the restriction is lifted, the terms of the classes also changes. Y could represent men and z could represent doctors but they aren’t restricted to that distinction since we are lifting the exclusiveness given to each class by the “+” sign. Furthermore, we don’t know how many of the men represented by y are doctors and how many of the doctors represented by z are men since they aren’t exclusive to one another. In other words, now they can be inclusive of one another where y and z can both represent men and doctors. However, we don’t know how many doctors and men are represented by each class. We just know they both represent both traits. Hence, we can’t deduce Boole’s equation (8) by being given x and z. We can get the total number of individuals in y but not specifically how many doctors and how many men are represented by y.

Task 10

Recall that for classes, the expression ‘1 + 1’ is meaningless for Boole. (Why?) Suppose we dropped Boole’s restriction on the use of ‘+’. Would it make sense to assign either of the two values (0 or 1) to the expression 1 + 1 as a statement about classes? Explain. Then consider the expression ‘1 + 1’ as a numerical statement within the ‘Algebra of Number;’ does it make sense to assign either 0 or 1 to the sum 1 + 1? Why or why not?

‘1 + 1’ is meaningless to Boole since Boole justified + as signifying the classification of mutually disjoint classes. However, in the expression ‘1+1’ we see that on both sides of the + are 1. If the restriction of the + holds, then 1 + 1 has no meaning. This is because they are both the same classes and violate the restriction by not being disjoint. They both share all the values of each other so they aren’t mutually disjoint.

Lifting the restriction of ‘+’ would mean that the classes on either side of the ‘+’ don’t have to be mutually disjoint. In that case, yes, it would make sense to assign either 0 or 1 or even both to either side of the +. However, what good would it do to combine any two classes that are exactly the same? We can assign either value 0 or 1 to either side if the value on either side is either 1 or 0 but not both. It wouldn’t make sense to have either of those on both sides of the expression since there wouldn’t be any distinction between the classes.

If the statement were the Algebra of Number then it wouldn’t matter if you assigned either 0 or 1 to either side of the expression since there is no rule restricting this assignment.

Task 12

In outline form, Boole’s proof of Proposition IV is as follows:

Given: x = x2.

Then x − x 2 = 0,

And x (1 − x) = 0

Interpreting the equation x (1 − x) = 0, we conclude that the classes x and 1 − x have no common members. Thus, it is impossible for “any being to possess a quality, and at the same time not to possess it.” Does Boole’s formal manipulation of the law x2 = x provide convincing proof that the Principle of Contradiction is a ‘consequence of a law of thought, mathematical in its form,’ rather than being an ‘axiomatic law’? Explain.

Boole’s manipulation of the law does provide convincing proof that the Principle of Contradiction is a ‘consequence of a law of thought, mathematical in its form.’ This is because the manipulation follows algorithmic steps that lead to a deduction in the end that expresses the conclusion. On the other hand, an axiomatic law is an established statement without contradiction of criticism. An axiomatic law is a self-evident, established, and accepted fact. The process show by Boole isn’t an axiomatic law since it isn’t an established obvious fact as he had to go through the process of proving it in an algorithmic and mathematical way, so it can be accepted as a ‘consequence of a law of thought.’

Task 14

Boole then gives several examples of how to use this rule, the first of which involves the following classes: x represents hard substances, y represents elastic substances, and z represents metals. Use Boole’s rules to express each of the following in these terms; remember Boole’s restriction on exception!

x = hard substances

y = elastic substances

z = metals

1. non-elastic metals

z(1-y)

1. hard substances, except metals

x(1-z)

1. Hard substances, except those which are metallic and non-elastic and those which are elastic and non-metallic.

x(1-z)(1-y) + xy(1-z)

1. Hard substances except those which are metallic and non-elastic, and substances which are elastic and non-metallic.

x(1-z)(1-y) + y(1-z)

1. metallic substances, except those which are neither hard nor elastic

z(1-x)(1-y)

Task 16

Let x to represent the class ‘men,’ and consider the correct logical principle which Venn described as follows:

“‘man’ and ‘not-man’, taken together, should comprise ‘all’, i.e. the logical universe

First represent this logical principle as an equation using Venn’s notation ‘ x ’ to designate the class ‘not-men.’ Then represent this same principle as an equation using Boole’s notation ‘1 − x’ to designate the class ‘not-men.’ Comment on the advantages and disadvantages of these two methods of representation.

Venn’s notation:

x + x = 1

Venn’s notation is much shorter and nice to look at. Defining not x in Venn’s notation looks much more succinct than in Boole’s notation. However, you’d need to know what x is to implement it in any representation. It might’ve been an unfamiliar notation to people if it were mentioned for the first time.

Boole’s notation:

X + (1-x) = 1

Boole’s representation makes the expression look like an algebraic expression that can be solved for x rather than a logical expression for all the elements in the universe. This representation also looks lengthier than Venn’s notation. However, the part to the right of the ‘+’ is easier to understand as the complement of x.

Task 18

1. Explain why the class which Boole expressed by p + (1 − p)r does include those things which are both productive of pleasure and preventative of pain. How would Boole represent this same class under the interpretation that things which are both productive of pleasure and preventative of pain are to be excluded? How would Venn have written this class under this same interpretation?

To the left of the + we see that Boole wrote just p. P represents the productive of pleasure. On this side of the expression he didn’t specify anything else then the productive of pleasure. So, we can go onto assume that Boole isn’t excluding the preventative of pain on the same side. Nonetheless, both productive of pleasure and preventative of pain are inclusive on the left side of the expression. However, on the right side of the expression Boole wrote (1-p)r which signifies preventative of pain but clearly excludes productive of pleasure. If either productive of pleasure or preventative of pain are excluded from the other Boole would represent it as:

p(1-r) + r(1-p)

This reads as productive of pleasure while not preventative of pain and preventative of pain while not productive of pleasure.

Venn would have written it as p + r where the expression would read either productive of pleasure or preventative of pain or both.

1. Compare the symbolic expressions for the two classes listed in (i) and (ii) below under both two interpretations of + (inclusive versus inclusive), first in Boole’s system and then in Venn’s system. Use the notation 1 − x to represent the class of not-x in both cases, and illustrate the sets in question by a Venn diagram.

w = wealth

t = things transferable

s = limited in supply

p = productive of pleasure

r = preventative of pain

1. things transferable or not limited in supply

Inclusive

Boole’s representation: w = t + s

Venn’s representation: w = t + s

Exclusive

Boole’s representation: w = t(1-s) + s(1-t)

Venn’s representation: w = t + s

1. things either not transferable or not productive of pleasure

Inclusive

Boole’s representation: w = (1-t) (1-(1-s)) + (1-s) (1-(1-t))

Venn’s representation: w = (1-t) + (1-s)

Exclusive

Boole’s representation: w = (1-t) (1-(1-s)) + (1-s) (1-(1-t))

Venn’s representation: w = (1-t) + (1-s)

1. Comment on the advantages of Venn’s system relative to those of Boole’s system in terms of economy or efficiency of expression.

Venn’s more compact representation of the ideas gives an expression that is much easier to read than Boole’s representation. Furthermore, Venn’s representation was more economic saving space and time. Venn’s expression only needs half of what you’d need to write for Boole’s representation.

Task 20

In other words, by multiplying the classes ‘x’ and ‘1 − y’ defined in this excerpt, Venn was then able to represent the class ‘clergy, except the teetotalers’ by the expression ‘x(1−y)’. Yet Venn intended this example to serve as an illustration of how ‘the process of subtraction, with the corresponding symbolic sign, can be dispensed with.’ Explain why there is no inconsistency here. Then indicate how the class ‘The clergy, except the teetotalers’ could be represented symbolically by Venn without using the symbol ‘−’ for any purpose.

X by itself only represents itself and nothing else. Just expressing the class x automatically excludes everything else unless specifically conjoined with x as in xy where both x and y are represented. However, when x is represented by itself then it excludes any other classes since there is no mention of them.

So, to represent the class ‘The clergy, except the teetotalers’ where the class clergy can be represented by x and the class teetotalers be represented by y we can just say ‘x’. This would represent the class clergy and automatically exclude the class teetotalers since the class teetotalers isn’t a part of the class clergy. We’d need to specifically label the class teetotalers in our expression to acknowledge their existence. In this expression ‘x’ the class clergy is acknowledged, but the class ‘y’ represented teetotalers is excluded without mention of it.

Task 22

Will any correction be needed in the case where we start with a difference instead of a sum? That is, if z = x − y in Venn’s system, can we always conclude that z + y = x? Why or why not? Sketch a Venn diagram to illustrate your reasoning, and compare this to the situation in Boole’s system. (Remember the restrictions which Boole and Venn placed upon subtraction.)

A correction will be needed in the case that we start with a difference instead of a sum. This is because we don’t know how many element of either of the classes x and y are shared by z. Since in Venn’s system the elements in the ‘+’ relation can be either, or, or both the distinction between which elements are in the ‘both’ category and which are in the ‘or’ isn’t quite clear. This will lead to error in determining the quantity of the contents of each set.

Task 24

As noted by Venn, Boole included the first of these two formula in his Laws of Thought (in a chapter we did not consider in this project). What conditions on A, B and x would be needed in order for the second formula to be acceptable in Boole’s system? Sketch a Venn diagram illustrating classes A, B and x which satisfy these conditions. Then explain why this second formula can be simplified to (A + x)(B + x) = A + B when these conditions are met.

(1) (A + x)(B + x) = AB + x

(2) (A + x)(B + x) = Ax + Bx

Both A and B need to be exclusive of each other where their elements don’t overlap. They also need to be exclusive in relation. We have to refer to either one or the other when related with the ‘+’ sign.